

Darboux Transformation for the Non-isospectral AKNS Hierarchy and Its Asymptotic Property

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Abstract

In this article, the Darboux transformation for the non-isospectral AKNS hierarchy is constructed. We show that the Darboux transformation for the non-isospectral AKNS hierarchy is not an auto-Bäcklund transformation, because the integral constants of the hierarchy will be changed after the transformation. The transform rule of the integral constants will be also derived. By this means, the soliton solutions of the nonlinear equations derived by the non-isospectral AKNS hierarchy can be found.

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1 Introduction

The AKNS hierarchy is one of the most important integrable systems. Many nonlinear equations, which involve many famous nonlinear differential equation, are equivalent to the integrability condition of AKNS hierarchy [7, 8, 11]. A unified approach to construct Darboux transformations with matrix form for AKNS hierarchy was founded by C.H. Gu [3–6, 9, 10]. This approach works for many soliton equations [4]. The method in these literatures can also be generalized to the non-isospectral AKNS hierarchy [15]. The Darboux transformation for the non-isospectral AKNS hierarchy has an essential difference from the standard case, that its integral constants are not conserved by the transform. So one cannot get the soliton solution of the relevant nonlinear equation by acting the

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Darboux transformation on the seed solution of its own [13–15]. In this article, one will see that the relation of the integral constants between the relevant non-isospectral AKNS hierarchies can be calculated through the asymptotic property of the elementary solution. Then the soliton solution of a certain differential equation can be found by acting the Darboux transformation on the seed solution of another equation.

2 Non-isospectral AKNS Hierarchy

The standard AKNS hierarchy is generalized by Ablowitz, Kaup, Newell and Segur [1] in 1974. Now we use the language in [12] to define it. We assume throughout that the matrix $J \in \mathfrak{sl}(N)$ in this article is fixed, diagonal, with distinct complex eigenvalues,

$$J = \text{diag}(J_1, \dots, J_N), \quad J_i \neq J_j \quad \text{if} \quad i \neq j, \quad (2.1)$$

and

$$\mathfrak{sl}(N)_J = \{X \in \mathfrak{sl}(N) : [J, X] = 0\} \quad (2.2)$$

$$\mathfrak{sl}(N)_J^\perp = \{Y \in \mathfrak{sl}(N) : \text{tr}(XY) = 0 \text{ for } X \in \mathfrak{sl}(N)_J\} \quad (2.3)$$

denote the centralizer of J and its orthogonal complement in $\mathfrak{sl}(N)$ respectively. One can easily verify the following facts.

Lemma 2.1. *$\mathfrak{sl}(N)$ has the direct sum decomposition $\mathfrak{sl}(N) = \mathfrak{sl}(N)_J \oplus \mathfrak{sl}(N)_J^\perp$ with respect to vector space.*

Lemma 2.2. *The matrix $P \in \mathfrak{sl}(N)_J^\perp$ if and only if the diagonal coefficients of P vanish.*

Lemma 2.3. *The mapping $\text{ad } J : \mathfrak{sl}(N) \rightarrow \mathfrak{sl}(N)_J^\perp$ is a homomorphism, and $\ker(\text{ad } J) = \mathfrak{sl}(N)_J$, which is equivalent to that the restriction of the mapping $\text{ad } J : \mathfrak{sl}(N)_J^\perp \rightarrow \mathfrak{sl}(N)_J^\perp$ is an isomorphism.*

We now turn to the non-isospectral AKNS hierarchy. Assume that

$$U(\lambda) = \lambda J + P, \quad V(\lambda) = \sum_{i=0}^n V_i \lambda^i, \quad (2.4)$$

where λ satisfies the scalar equation

$$\lambda_t = \sum_{i=0}^n f_i \lambda^i \quad (2.5)$$

and $P \in \mathfrak{sl}(N)_J^\perp$, $V_i \in \mathfrak{sl}(N)$ are matrices independent of λ . The coupled $N \times N$ matrix equations

$$\begin{cases} \Phi_x = U(\lambda)\Phi \\ \Phi_t = V(\lambda)\Phi \end{cases} \quad (2.6)$$

are called AKNS system and the system is integrable if and only if the zero curvature condition

$$U_t - V_x + [U, V] = 0 \quad (2.7)$$

holds. Comparing the coefficients of λ in (2.7) leads to the following equations

$$[J, V_n] = 0, \quad (2.8)$$

$$f_i J - V_{i,x} + [J, V_{i-1}] + [P, V_i] = 0 \quad (1 \leq i \leq n), \quad (2.9)$$

$$f_0 J + P_t - V_{0,x} + [P, V_0] = 0. \quad (2.10)$$

Define

$$V_i^{diag} = \pi_0(V_i), \quad V_i^{off} = \pi_1(V_i), \quad (2.11)$$

where $\pi_0, \pi_1 \in \text{End}(sl(N))$ denote the projection of $sl(N)$ onto $sl(N)_J$ and $sl(N)_J^\perp$ respectively, then one can find the following recurrence formulae

$$V_n^{off} = (\text{ad } J)^{-1} 0 = 0, \quad (2.12)$$

$$V_{i,x}^{diag} = f_i J + \pi_0 \left([P, V_i^{off}] \right) \quad (0 \leq i \leq n), \quad (2.13)$$

$$V_i^{off} = (\text{ad } J)^{-1} \left(V_{i+1,x}^{off} - \pi_1([P, V_{i+1}]) \right) \quad (0 \leq i \leq n-1). \quad (2.14)$$

and the nonlinear PDE on P

$$P_t - V_{0,x}^{off} + [P, V_0^{diag}] = 0. \quad (2.15)$$

If P satisfies the asymptotic condition

$$\lim_{x \rightarrow \infty} |x|^k \partial_x^m (P(x, t)) = 0 \quad \text{for } t \text{ and nonnegative integer } k, m \quad (2.16)$$

i.e. $P(t, \cdot)$ is in the Schwartz class which is denoted by $\mathcal{S}(\mathbb{R}, sl(N)_J^\perp)$, then V_i will be determined uniformly up to n integral constants $\alpha_i(t) \in sl(N)$. In fact, $V_i^{diag}(t)$ can be defined by

$$V_i^{diag} = \alpha_i(t) + f_i J x + \int_{-\infty}^x \pi_0 \left([P, V_i^{off}] \right) dx. \quad (2.17)$$

Moreover, we have the following conclusion.

Theorem 2.4. *For t and nonnegative integer k, m ,*

$$\lim_{x \rightarrow -\infty} |x|^k \partial_x^m (V_i - f_i J x - \alpha_i(t)) = 0 \quad (0 \leq i \leq n), \quad (2.18)$$

and especially

$$\lim_{x \rightarrow -\infty} (V - \sum_{i=0}^n f_i J x \lambda^i) = \sum_{i=0}^n \alpha_i(t) \lambda^i. \quad (2.19)$$

Proof. Assume inductively that (2.18) holds for i , then

$$\begin{aligned} & \lim_{x \rightarrow -\infty} |x|^k \partial_x^m ([P, V_i]) \\ &= \lim_{x \rightarrow -\infty} |x|^k \partial_x^m ([P, V_i - f_i Jx - \alpha_i(t)] + [P, f_i Jx + \alpha_i(t)]) = 0. \end{aligned} \quad (2.20)$$

It follows that

$$\lim_{x \rightarrow -\infty} |x|^k \partial_x^m V_{i-1}^{off} = (\text{ad } J)^{-1} \lim_{x \rightarrow -\infty} |x|^k \partial_x^m (V_{i,x}^{off} - \pi_1([P, V_i])) = 0. \quad (2.21)$$

So V_{i-1}^{diag} can be defined by

$$V_{i-1}^{diag} = \alpha_{i-1}(t) + f_{i-1} Jx + \int_{-\infty}^x \pi_0([P, V_{i-1}^{off}]) dx, \quad (2.22)$$

and (2.18) holds for $i - 1$. \square

Remark 2.1. If the polynomial $f(\lambda)$ degenerates to vanish, the relevant AKNS hierarchy is called *isospectral*, otherwise we called it *non-isospectral*. That is to say the non-isospectral case is a generalization of the standard case.

3 Darboux Transformation for the Non-isospectral AKNS Hierarchy

The Darboux transformation for the non-isospectral problem with 2×2 matrix coefficients is constructed by C. Rogers and W.K. Schief [11]. Their method can be generalize to $N \times N$ case [15]. We states the conclusion here.

Theorem 3.1. *Consider the $N \times N$ matrix equation*

$$\Phi_t = V(\lambda)\Phi, \quad V(\lambda) \in sl(N) \quad (3.1)$$

where

$$V(\lambda) = \sum_{i=0}^n V_i \lambda^i. \quad (3.2)$$

The spectral parameter λ satisfies the scalar differential equation

$$\lambda_t = f(\lambda) = \sum_{i=0}^{n+2} f_i \lambda^i \quad (3.3)$$

where $f(\lambda)$ is a polynomial of λ , the highest power of λ in $f(\lambda)$ is not more than $n + 2$. Let h_1, \dots, h_N be the known vector-valued eigenfunctions of the equation (3.1) corresponding to the parameters $\lambda_1, \dots, \lambda_N$ and at least two of them are different. Set

$$S = H \Lambda H^{-1}, \quad P(\lambda) = \lambda I - S \quad (3.4)$$

where

$$H = (h_1, \dots, h_N), \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_N). \quad (3.5)$$

Then, the transformation

$$\Phi' = D(\lambda)\Phi = p(\lambda)P(\lambda)\Phi \quad (3.6)$$

$$V'(\lambda) = P(\lambda)V(\lambda)P^{-1}(\lambda) + \frac{dP(\lambda)}{dt}P^{-1}(\lambda) + \frac{dp(\lambda)}{dt}p^{-1}(\lambda) \quad (3.7)$$

where

$$p(\lambda)^{-N} = \det P(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_N) \quad (3.8)$$

is a Darboux transformation for (3.1), i.e. the following conditions are satisfied.

(i) $V'(\lambda) \in \mathfrak{sl}(N)$.

(ii) $V'(\lambda)$ has the same polynomial structure as $V(\lambda)$.

According to Theorem 3.1, $V'(\lambda)$ satisfies the equation

$$V'(\lambda)(\lambda I - S) = (\lambda I - S)V(\lambda) + (f(\lambda)I - S_t) - g(\lambda)(\lambda I - S), \quad (3.9)$$

where

$$g(\lambda) = \frac{1}{N} \sum_{i=1}^N \frac{f(\lambda) - f(\lambda_i)}{\lambda - \lambda_i} = \sum_{i=0}^{n+1} g_i \lambda^i. \quad (3.10)$$

Comparing the coefficients of λ leads to

$$\begin{aligned} V'_n &= V_n + (f_{n+1} - g_n) + g_{n+1}S = V_n + f_{n+2}S + f_{n+1} - g_n, \\ V'_{n-j} &= V_{n-j} + V'_{n-j+1}S - SV_{n-j+1} + (f_{n-j+1} - g_{n-j}) + g_{n-j+1}S \\ &= V_{n-j} + \sum_{k=1}^j [V_{n-j+k}, S] S^{k-1} + \sum_{k=0}^{j+1} f_{n-j+k+1} S^k - g_{n-j}. \end{aligned} \quad (3.11)$$

Such Darboux transformation works evidently for the non-isospectral AKNS hierarchy, nevertheless the integral constants of $V'(\lambda)$ are generally different from the ones of $V(\lambda)$. The asymptotic property of D is necessary to determine the relation between the two sets of the integral constants. To state the asymptotic property of D , we need part of the scattering theory of Beals and Coifman [2]. Let

$$\Gamma_J = \{\zeta \in \mathbb{C} : \text{Re}(\zeta(J_j - J_k)) = 0, 1 \leq j \leq k \leq N\}. \quad (3.12)$$

Theorem 3.2 (Beals-Coifman). *If $P \in \mathcal{S}(\mathbb{R}, \mathfrak{sl}(N)_J^\perp)$, then for $\lambda \in \mathbb{C} \setminus \Gamma_J$, there exist elementary solutions $\Phi_l(x, \lambda)$ and $\Phi_r(x, \lambda)$ of the differential equation*

$$\Phi_x = (\lambda J + P)\Phi, \quad (3.13)$$

which satisfy

$$\Phi_l(x, \lambda) = O(\exp(\lambda Jx)) \quad x \rightarrow -\infty \quad (3.14)$$

$$\Phi_r(x, \lambda) = O(\exp(\lambda Jx)) \quad x \rightarrow +\infty \quad (3.15)$$

Now one can construct a Darboux transformation for non-isospectral AKNS hierarchy in the following way. Assume that $J_1 > J_2 > \cdots > J_N$ without losing generality. Firstly, set

$$\lambda_i(0) \in \mathbb{C} \setminus \Gamma_J, \quad \text{for } 1 \leq i \leq N, \quad (3.16)$$

then there exists an interval $(-t_0, t_0)$, such that $\lambda_i(t)$ solves (3.3) and $\lambda_i(t) \in \mathbb{C} \setminus \Gamma_J$. Then the eigenfunction $h_i(t, x)$ solving the integrable non-isospectral AKNS hierarchy can be denoted by

$$h_i(t, x) = \Phi_l(x, \lambda_i(t))L_i(t) = \Phi_r(x, \lambda_0(t))R_i(t). \quad (3.17)$$

According to Theorem 3.2, one can easily find that

$$\det H = \sum_{\sigma \in S_N} a_\sigma(t) O \left(\exp \left(\sum_{k=0}^N \lambda_k(0) J_{\sigma(k)} \right) x \right), \quad (3.18)$$

where $H = (h_1, \dots, h_N)$ and S_N denotes permutation group with order N . Let

$$\begin{aligned} m &= \min \left\{ \sum_{k=0}^N \lambda_k(0) J_{\sigma(k)} : \sigma \in S_N \right\}, \\ M &= \max \left\{ \sum_{k=0}^N \lambda_k(0) J_{\sigma(k)} : \sigma \in S_N \right\}, \end{aligned} \quad (3.19)$$

then

$$\begin{aligned} \det H &= O(\exp mx) & x \rightarrow -\infty, \\ \det H &= O(\exp Mx) & x \rightarrow +\infty. \end{aligned} \quad (3.20)$$

if the relevant coefficients $a_\sigma(t)$ in (3.18) do not vanish. Secondly, choose

$$\lambda_1(0) = \cdots = \lambda_{k_0}(0) < 0, \lambda_{k_0+1}(0) = \cdots = \lambda_N(0) > 0, \quad (3.21)$$

and carefully select h_1, \dots, h_N such that (3.20) holds, then we can prove that

$$\lim_{x \rightarrow -\infty} S = \lim_{x \rightarrow -\infty} H \Lambda H^{-1} = \Lambda, \quad \text{for } t \in (-t_0, t_0). \quad (3.22)$$

and $\lim_{x \rightarrow +\infty} S$ is also a diagonal matrix similar to Λ , which leads to

$$P'(t, \cdot) = P + [J, S] \in \mathcal{S}(\mathbb{R}, sl(N)_{\frac{1}{J}}^{\perp}), \quad \text{for } t \in (-t_0, t_0). \quad (3.23)$$

(One may admit the conclusion for the moment, and we will prove it in the next section.) Setting $x \rightarrow -\infty$, with the above property, (3.11) leads to that

$$\begin{aligned} \alpha'_n(t) &= \alpha_n(t) + f_{n+2}\Lambda + (f_{n+1} - g_n), \\ \alpha'_{n-j}(t) &= \alpha_{n-j}(t) + \sum_{k=0}^{j+1} f_{n-j+k+1}\Lambda^k - g_{n-j} \quad (j \geq 0). \end{aligned} \quad (3.24)$$

If we set

$$\beta_j(\Lambda) = \sum_{k=0}^{n-j+1} f_{j+k+1} \Lambda^k - g_j \quad (0 \leq j \leq n), \quad (3.25)$$

the solution of the nonlinear PDE via non-isospectral AKNS hierarchy with the integral constants $\alpha_j(t)$ can be attained by acting the Darboux transformation on the seed solution of the equation of the hierarchy with the integral constants $\alpha_j(t) - \beta_j(\Lambda)$.

4 Proof of the Asymptotic Property

Firstly, we state an elementary inequality.

Lemma 4.1. *Assume that $x_i, y_i \in \mathbb{R}$ ($1 \leq i \leq N$), $x_1 \leq x_2 \leq \dots \leq x_N$ and*

$$M = \max \left\{ \sum_{i=1}^N x_i y_{\sigma(i)} : \sigma \in S_N \right\}, \quad m = \min \left\{ \sum_{i=1}^N x_i y_{\sigma(i)} : \sigma \in S_N \right\} \quad (4.1)$$

then

$$\begin{aligned} \sum_{i=1}^N x_i y_{\sigma(i)} = M & \quad \text{if and only if} \quad y_{\sigma(1)} \leq y_{\sigma(2)} \leq \dots \leq y_{\sigma(N)}, \\ \sum_{i=1}^N x_i y_{\sigma(i)} = m & \quad \text{if and only if} \quad y_{\sigma(1)} \geq y_{\sigma(2)} \geq \dots \geq y_{\sigma(N)}. \end{aligned} \quad (4.2)$$

Let $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_n\}$ be finite sets which satisfy $\#X = \#Y$, and define

$$\begin{aligned} \langle X, Y \rangle_{\max} &= \max \left\{ \sum_{i=1}^N x_i y_{\sigma(i)} : \sigma \in S_N \right\}, \\ \langle X, Y \rangle_{\min} &= \min \left\{ \sum_{i=1}^N x_i y_{\sigma(i)} : \sigma \in S_N \right\}. \end{aligned} \quad (4.3)$$

Then one can see that the coefficients of H and its adjoint H^* satisfy

$$H_{ij} = O(\exp(\lambda_j J_i x)), \quad H_{ij}^* = O(\exp(\langle \Lambda \setminus \lambda_j, J \setminus J_i \rangle_{\min} x)), \quad x \rightarrow -\infty, \quad (4.4)$$

which implies that

$$H_{ik} H_{jk}^* = O(\exp(\lambda_k J_i + \langle \Lambda \setminus \lambda_k, J \setminus J_j \rangle_{\min} x)). \quad (4.5)$$

Now, we begin to prove (3.22), which is equivalent to

$$\lim_{x \rightarrow -\infty} \text{ent}_{ij}((\det H)^{-1} H \Lambda H^*) = \delta_{ij} \lambda_j, \quad (4.6)$$

where $J = \{J_1, \dots, J_N\}$ and $\Lambda = \{\lambda_1, \dots, \lambda_N\}$ are regarded as two sets with N elements. We prove it in the following cases.

Case 1, $i < j$. Noting $J_i > J_j$, it implies that

$$\lambda_k J_i + \langle \Lambda \setminus \lambda_k, J \setminus J_j \rangle_{\min} > \lambda_k J_j + \langle \Lambda \setminus \lambda_k, J \setminus J_j \rangle_{\min} \geq m, \quad k > k_0. \quad (4.7)$$

So it follows from

$$\sum_{k=0}^N H_{ik} H_{jk}^* = 0, \quad (4.8)$$

that

$$\begin{aligned} \lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\sum_{k \leq k_0} H_{ik} H_{jk}^* \right) &= - \lim_{x \rightarrow -\infty} \exp(-mx) \left(\sum_{k > k_0} H_{ik} H_{jk}^* \right) \\ &= - \lim_{x \rightarrow -\infty} \exp(-mx) \left(\sum_{k > k_0} O(\exp(\lambda_k J_i + \langle \Lambda \setminus \lambda_k, J \setminus J_j \rangle_{\min}) x) \right) = 0 \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} \lim_{x \rightarrow -\infty} \text{ent}_{ij} S &= \lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\sum_{k=1}^N \lambda_k H_{ik} H_{jk}^* \right) \\ &= \lim_{x \rightarrow -\infty} \exp(-mx) \left(\lambda_1 \sum_{k \leq k_0} H_{ik} H_{jk}^* + \lambda_N \sum_{k > k_0} H_{ik} H_{jk}^* \right) = 0. \end{aligned} \quad (4.10)$$

Case 2, $i > j$. It is similar to case 1 that $J_i < J_j$ leads to that

$$\lambda_k J_i + \langle \Lambda \setminus \lambda_k, J \setminus J_j \rangle_{\min} > \lambda_k J_j + \langle \Lambda \setminus \lambda_k, J \setminus J_j \rangle_{\min} \geq m, \quad k \leq k_0. \quad (4.11)$$

Then one can find that

$$\lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\sum_{k > k_0} H_{ik} H_{jk}^* \right) = - \lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\sum_{k \leq k_0} H_{ik} H_{jk}^* \right) = 0, \quad (4.12)$$

which implies

$$\lim_{x \rightarrow -\infty} \text{ent}_{ij} S = 0. \quad (4.13)$$

Case 3, $i = j \leq k_0$. It follows from Lemma 4.1 that

$$\lambda_k J_i + \langle \Lambda \setminus \lambda_k, J \setminus J_i \rangle_{\min} \geq m, \quad k > k_0, \quad (4.14)$$

which implies that

$$\begin{aligned} &\lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\sum_{k \leq k_0} H_{ik} H_{ik}^* \right) \\ &= \lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\sum_{k=1}^n H_{ik} H_{ik}^* - \sum_{k > k_0} H_{ik} H_{ik}^* \right) \\ &= \lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\det H - \sum_{k > k_0} H_{ik} H_{ik}^* \right) = 1. \end{aligned} \quad (4.15)$$

Hence

$$\lim_{x \rightarrow -\infty} \text{ent}_{ii} S = \lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\lambda_1 \sum_{k \leq k_0} H_{ik} H_{ik}^* + \lambda_N \sum_{k > k_0} H_{ik} H_{ik}^* \right) = \lambda_1. \quad (4.16)$$

Case 4, $i = j > k_0$. It is similar to case 3 that one can find

$$\lim_{x \rightarrow -\infty} \text{ent}_{ii} S = \lim_{x \rightarrow -\infty} (\det H)^{-1} \left(\lambda_1 \sum_{k \leq k_0} H_{ik} H_{ik}^* + \lambda_N \sum_{k > k_0} H_{ik} H_{ik}^* \right) = \lambda_N. \quad (4.17)$$

That $\lim_{x \rightarrow +\infty} S$ is a diagonal matrix similar to Λ can be proved by the same means. Combined with the asymptotic property of the elementary solution Φ , one can find (3.23) evidently.

5 The Soliton Solution of the Non-isospectral MKdV Equation

The following equation

$$u_t + (1 - \frac{1}{4}x)(u_{xxx} + 6u^2u_x) - \frac{3}{4}u_{xx} - u^3 - \frac{1}{2}u_x \int_{-\infty}^x u^2 dx = 0, \quad (5.1)$$

compared with the standard MKdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0, \quad (5.2)$$

is called the non-isospectral MKdV equation, which is equivalent to the zero curvature condition of the non-isospectral AKNS system defined by

$$n = 3, J = \text{diag}(1, -1), p = -q = u(t, x), \alpha_3 = -4J, \alpha_0 = \alpha_1 = \alpha_2 = 0.$$

and $f(\lambda) = \lambda^3$. Directly calculation leads to $\lambda^2 = (\kappa - 2t)^{-1}$ ($\kappa \in \mathbb{R}$). Choose

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix} = \begin{pmatrix} -(\kappa_0 - 2t)^{-\frac{1}{2}} & 0 \\ 0 & (\kappa_0 - 2t)^{-\frac{1}{2}} \end{pmatrix}, \quad (5.3)$$

then one can find

$$g(\lambda) = \lambda^2 + \lambda_0^2. \quad (5.4)$$

According to (3.25), $\beta(\Lambda)$ should be defined by

$$\begin{aligned} \beta_3(\Lambda) &= 0, & \beta_2(\Lambda) &= f_3 - g_2 = 0, \\ \beta_1(\Lambda) &= f_3\Lambda = \Lambda, & \beta_0(\Lambda) &= f_3\Lambda^2 - g_0 = 0. \end{aligned} \quad (5.5)$$

Hence the soliton solution of (5.1) can be attained by acting the Darboux transformation on the nonisospectral AKNS system with the integral constants

$\alpha'(t) = \alpha(t) - \beta(\Lambda)$, i.e. $\alpha'_3(t) = -4J$, $\alpha'_2 = \alpha'_0 = 0$, $\alpha'_1 = -\Lambda$. Choose the trivial solution for the seed solution, i.e. set $P = 0$, then the matrices U, V_i with the integral constants $\alpha'(t)$ are defined by the following

$$\begin{aligned} U &= \begin{pmatrix} (\kappa - 2t)^{-\frac{1}{2}} & 0 \\ 0 & -(\kappa - 2t)^{-\frac{1}{2}} \end{pmatrix} \\ V_3 &= \begin{pmatrix} x - 4 & 0 \\ 0 & -x + 4 \end{pmatrix}, \quad V_2 = 0, \\ V_1 = -\Lambda &= \begin{pmatrix} (\kappa_0 - 2t)^{-\frac{1}{2}} & 0 \\ 0 & -(\kappa_0 - 2t)^{-\frac{1}{2}} \end{pmatrix}, \quad V_0 = 0, \end{aligned} \quad (5.6)$$

and the relevant elementary solution is

$$\Phi(x, t, \lambda) = \begin{pmatrix} C(t) \exp(\lambda x) & 0 \\ 0 & (C(t))^{-1} \exp(-\lambda x) \end{pmatrix}, \quad (5.7)$$

where

$$C(t) = \exp \left(- \int_0^t (4\lambda^3 + \lambda_0 \lambda) dt \right). \quad (5.8)$$

Set

$$H = \begin{pmatrix} C_0(t) \exp(\lambda_0 x) & -(C_0(t))^{-1} \exp(-\lambda_0 x) \\ (C_0(t))^{-1} \exp(-\lambda_0 x) & C_0(t) \exp(\lambda_0 x) \end{pmatrix}, \quad (5.9)$$

where

$$C_0(t) = \exp \left(- \int_0^t (4\lambda_0^3 + \lambda_0^2) dt \right) = \exp(-4\lambda_0 - \ln(-\lambda_0) + c_0) \quad (5.10)$$

and $c_0 = (4\lambda_0 + \ln(-\lambda_0))|_{t=0}$, then

$$S = H \Lambda H^{-1} = \lambda_0 \begin{pmatrix} \tanh 2\xi & \operatorname{sech} 2\xi \\ \operatorname{sech} 2\xi & -\tanh 2\xi \end{pmatrix}, \quad (5.11)$$

where $\xi = \lambda_0 x - 4\lambda_0 - \ln(-\lambda_0) + c_0$. Following from $P' = P + [J, S]$, one can see the 1-soliton solution of (5.1)

$$u = 2\lambda_0 \operatorname{sech} 2\xi. \quad (5.12)$$

By this means, the 2-soliton solution can also be attained.

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